Dispersion relations with finite propagation speed

1980 J. Phys. A: Math. Gen. 13 2615

(http://iopscience.iop.org/0305-4470/13/8/010)

J. Phys. A: Math. Gen. **13** (1980) 2615–2621. Printed in Great Britain

Dispersion relations with finite propagation speed

Yossi Avron

Department of Physics, Technion—Israel Institute of Technology, Haifa, Israel

Received 8 May 1979, in final form 29 January 1980

Abstract. We consider waves generated at $t=0$ by prescribing Cauchy data. It is shown that, under weak assumptions the waves propagate inside the forward light cone only if $\Sigma_j \omega_j^2(\boldsymbol{k})$ is a polynomial in $\boldsymbol{k}$ of degree 2.

1. Introduction

It is well known that causality imposes strong restrictions on the admissible functions for the index of refraction, $n(\omega)$, and the dielectric function $\epsilon(\omega)$. This was first realised by Sommerfeld in 1914 and subsequently by Brillouin (Brillouin 1960). Among the numerous implications of these results are the Kramers–Kroning relations that relate the real and imaginary parts of $n(\omega)$ (or $\epsilon(\omega)$) through Cauchy integral formulae. The purpose of the present paper is to perform a similar analysis for the dispersion function $\omega(\boldsymbol{k})$. We shall consider waves propagating in a linear and homogeneous medium. By linearity we mean that the superposition principle holds, i.e. if $U_1(\boldsymbol{x}, t)$ and $U_2(\boldsymbol{x}, t)$ solve the relevant wave equation so does $U_1(\boldsymbol{x}, t)+U_2(\boldsymbol{x}, t)$. By homogeneity we mean that if $U(\boldsymbol{x}, t)$ is a solution so is $U(\boldsymbol{x}+\boldsymbol{a}, t)$ for all $\boldsymbol{a}$. It will not be necessary to assume that the medium is isotropic, so $U(R\boldsymbol{x}, t)$ need not be a solution if $U(\boldsymbol{x}, t)$ is where R is a rotation. We shall also not assume that the wave equation is a partial differential equation.

Two model equations for the wave $U(\boldsymbol{x}, t)$ that satisfies the above assumptions are

$$\text{(a)} \qquad [\omega(\mathrm{i}\boldsymbol{\nabla})+\mathrm{i}\,\partial_t]U(\boldsymbol{x}, t)=0$$

$$\text{(b)} \qquad [\omega^2(\mathrm{i}\boldsymbol{\nabla})-\partial_{tt}]U(\boldsymbol{x}, t)=0.$$

In applications, the wave equations are often second order in time (at least in classical physics). This makes (b) the more interesting of the two. (a) is considered because the analysis is essentially the same but less involved. (The usual wave equation obtains for $\omega^2(\boldsymbol{x})=c^2x^2$.)

We shall assume that the wave is generated at $t=0$ by prescribing Cauchy data. This means that at $t=0$ $U(\boldsymbol{x}, 0)$ (and in case (b) also $\partial_t U(\boldsymbol{x}, 0)$) is specified. For $t<0$, $U(\boldsymbol{x}, t)$ is identically zero.

Our original motivation was to characterise the class of dispersion functions $\omega(\boldsymbol{k})$ that propagate the Cauchy data within the forward light cone and in a second stage, to see how this class accommodates classical dispersions such as phonons, capillary waves, plasma waves, electromagnetic waves in dielectric etc.

0305-4470/80/082615+07$01.50

As we shall show, waves generated at $t=0$ propagate inside the (forward) light cone only for $\Sigma_j \omega_j^2(\boldsymbol{k})$ polynomial in $\boldsymbol{k}$ of degree 2. Thus, apart from a few exceptions, the aforementioned examples cannot be accommodated with the requirement of finite speed of propagation.

It is important to realise that the causality criteria for $\omega(\boldsymbol{k})$ are neither implied by, nor in conflict with, the Sommerfeld and Brillouin causality criteria for $n(\omega)$. We shall discuss this point in some detail.

The problem treated by Sommerfeld and Brillouin and the problem treated here are different. Sommerfeld and Brillouin consider a semi-infinite, one-dimensional *half space* $x \geqslant 0$ with time evolution given for all $-\infty < t < \infty$ by

$$U(x, t) = \int_{-\infty}^{\infty} \tilde{A}(\omega)\, \mathrm{e}^{\mathrm{i}\omega[n(\omega)x/c - t]}\, \mathrm{d}\omega.$$

$U(x, t)$ is the wave and $\tilde{A}(\omega)$ is related to a source that excites the medium for $t \geqslant 0$.

A straightforward consequence of the Cauchy formula is that a sufficient condition for causality is

(a) $n(\omega)$ is analytic in the upper half ω plane;

(b) $\operatorname{Im} \omega(n(\omega)-1) \geqslant 0$ for $\operatorname{Im} \omega \geqslant 0$ as $|\omega| \to \infty$.

In contrast, the problem considered here has the roles of space and time interchanged. Typically, we consider the *entire axis* $-\infty < x < \infty$. A wave is excited at $t=0$ and for $t \geqslant 0$ *alone*; its time evolution is given by

$$U(x, t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \tilde{h}(k)\, \mathrm{e}^{\mathrm{i}[kx - \omega(k)t]}\, \mathrm{d}k. \qquad (*)$$

For $t<0$, $U=0$ identically. $\tilde{h}(k)$ is related to $U(x, t=0)$. As we shall show, $U(x, t)$ vanishes outside the forward light cone provided

$$\omega(k) = ak + b.$$

Thus, causality imposes a stronger condition on $\omega(k)$ than it does on $n(\omega)$. It is perhaps unexpected that the slight change of roles played by x and t in the two problems has such dramatic consequences. How this comes about should become clearer in what follows. Here we remark that the light cone condition distinguishes the time axis and an x–t symmetry is not to be expected. Indeed, that $n(\omega)$ is analytic for $\operatorname{Im} \omega \geqslant 0$ while $\omega(k)$ is entire for *all* k (with positive or negative imaginary parts) is related to the light cone containing arbitrary x's but only positive times.

There is also a direct way to see that the Sommerfeld–Brillouin results for $n(\omega)$ do not translate to the present problem.

For an isotropic medium $k = \omega n(\omega)$. However $k(\omega)$ cannot, in general, be inverted to yield a *unique* $\omega(k)$. A typical example is

$$n^2(\omega) = 1 + \omega_p^2/(\omega_0^2 - \omega^2 - \mathrm{i}\omega\Gamma)$$

for which $\omega(k)$ has four Riemann sheets. Furthermore, even if a branch is chosen arbitrarily, it will not propagate $(*)$ inside the light cone. This is easily seen since $\omega(k)$ has singularities both in the upper and lower k planes and for $|x| \geqslant ct$ the contour of integration in $(*)$ cannot be shifted to infinity (it is 'pinched' by the singularities). There is therefore no reason for $(*)$ to vanish.

2. Statement of results

Consider wave propagation according to the dynamics

$$\prod_{j=1}^{m} [\omega_j(-\mathrm{i}\nabla)+\mathrm{i}\,\partial_t]U(\boldsymbol{x}, t)=0, \qquad t\geqslant 0, \tag{1}$$

or equivalently

$$U(\boldsymbol{x}, t)=\begin{cases}(2\pi)^{-n/2}\sum\int \mathrm{d}^n k\, \tilde{h}_j(k)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{x}-\mathrm{i}\omega_j(k)t}, & t\geqslant 0,\\ 0, & t<0.\end{cases} \tag{1$'$}$$

m is the order in time of the differential equation. We shall consider only $m=1$ and $m=2$. $m=2$ is the physically more interesting case. m is also the number of 'branches' of the dispersion, i.e. the number of different $\{\omega_j(k)\}$. Indeed, for solvability of the $\{\tilde{h}_j(\boldsymbol{k})\}$ in terms of the Cauchy data $\{\partial^j U(\boldsymbol{x}, t=0)/\partial t^j\}$, $\omega_i(\boldsymbol{k})\neq\omega_j(\boldsymbol{k})$, $i\neq j$ (Krotschek and Knudt 1978). Equation (1) is a pseudodifferential equation: for light propagation in a vacuum, $\omega_{1,2}=\pm|\boldsymbol{k}|$, it reduces to an ordinary partial differential equation. We shall assume $\operatorname{Im}\omega_j(\boldsymbol{k})\leqslant M<\infty$ for all real $\boldsymbol{k}$, and M an arbitrary constant. This is a requirement of at most exponential gain of the medium. A tilde ($\sim$) denotes the Fourier transform.

Theorem 1. Let $\operatorname{Im}\omega(\boldsymbol{k})\leqslant M$, $\boldsymbol{k}$ real, $m=1$. Causality for *any* $\tilde{h}(\boldsymbol{k})$ implies

$$\omega(\boldsymbol{k})=\boldsymbol{a}\,.\,\boldsymbol{k}+b \tag{2}$$

for $\boldsymbol{a}$ real and $|\boldsymbol{a}|\leqslant c$.

Remarks.

(1) c is the velocity of light.

(2) *Any* stands in contradistinction to *all.*

(3) This is known as a no-go theorem in relativistic quantum mechanics (Velo and Wightman 1978).

Theorem 1 has a corollary that seems worth separating out:

Corollary 2. If, under the conditions of theorem 1, $U(\boldsymbol{x}, t)$ has support inside some fixed ball for all t, then $\omega(\boldsymbol{k})=$ constant.

This can be described as absence of (non-trivial) soliton solutions for the linear dynamics (1).

$m=2$ with $\omega_1(\boldsymbol{k})\neq\omega_2(\boldsymbol{k})$ (a.e.) is associated with a second-order differential equation. It is then natural to write (1) in terms of the initial data $f(\boldsymbol{x})\equiv U(\boldsymbol{x}, t=0)$ and $g(\boldsymbol{x})\equiv\partial_t U(\boldsymbol{x}, t=0)$:

$$U(\boldsymbol{x}, t)=(2\pi)^{-n/2}\int \mathrm{d}^n k\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{x}}\tilde{f}(\boldsymbol{k})\frac{\mathrm{e}^{-\mathrm{i}\omega_1(\boldsymbol{k})t}\omega_2(\boldsymbol{k})-\mathrm{e}^{-\mathrm{i}\omega_2(\boldsymbol{k})t}\omega_1(\boldsymbol{k})}{\omega_2(\boldsymbol{k})-\omega_1(\boldsymbol{k})}$$

$$-(2\pi)^{-n/2}\mathrm{i}\int \mathrm{d}^n k\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}.\boldsymbol{x}}\tilde{g}(\boldsymbol{k})\frac{\mathrm{e}^{-\mathrm{i}\omega_1(\boldsymbol{k})t}-\mathrm{e}^{-\mathrm{i}\omega_2(\boldsymbol{k})t}}{\omega_2(\boldsymbol{k})-\omega_1(\boldsymbol{k})}. \tag{3}$$

We shall say that $\boldsymbol{k}$ is a double point if $\omega_1(\boldsymbol{k})=\omega_2(\boldsymbol{k})$.

Theorem 3. Let $\omega_1(\boldsymbol{k}) \neq \omega_2(\boldsymbol{k})$ (a.e.), $\operatorname{Im}\omega_i(\boldsymbol{k}) \leq M$, $\boldsymbol{k}$ real, $i \in \{1, 2\}$. Let $(f, 0)$ (or $(0, g)$) be the (measurable) initial data with support inside a finite ball and Fourier transform which does not vanish on double points. If for such initial data $U(\boldsymbol{x}, t)$ propagates within the light cone of $f(g)$, then

(a) $\omega_1(\boldsymbol{k}) + \omega_2(\boldsymbol{k}) = \boldsymbol{a} \cdot \boldsymbol{k} + b$, $\boldsymbol{a}$ real;

(b) $\Sigma_j \omega_j^2(\boldsymbol{k})$ is a polynomial of degree ≤ 2 in $\boldsymbol{k}$.

Remarks.

(1) A class of causal dispersions is $\omega^2(\boldsymbol{k}) = a^2k^2 + b^2$, $0 \leq a \leq c$, b real. $a = c$, $b = 0$ describes light in a vacuum; $0 \leq a \leq c$, $b \neq 0$ describes certain non-dissipating plasma modes.

(2) In the isotropic situation $\boldsymbol{k}$ may be taken as a single complex variable and double points are (generically) discrete. It is then easy to arrange for suitable, non-vanishing initial data. In any case, δ-function initial data have a constant Fourier transform and so fulfil the requirements of the theorem. See also remark (2) of proposition 3.4.

It follows from theorems 1 and 3 that non-trivial attenuation is incompatible with causality for most initial data. This is in marked contrast with the corresponding boundary value problem which has non-trivial refraction index functions (Brillouin 1960, Nussenzweig 1972).

3. Proofs

Introduce the notation

$\Gamma = \{\boldsymbol{x}, t | t \geq 0, |x| \leq ct\}$: the forward light cone

$\Gamma + t_0 = \{\boldsymbol{x}, t | \boldsymbol{x}, (t - t_0) \in \Gamma\}$: the shifted cone

$-\Gamma = \{\boldsymbol{x}, t | (-\boldsymbol{x}, -t) \in \Gamma\}$: the backward light cone

$\Gamma^* = \{\boldsymbol{k}, \Omega | \Omega \geq 0, |\boldsymbol{k}| \leq \Omega/c\}$: the dual cone

$z = z_1 + \mathrm{i}z_2$

$\tilde{U}(\boldsymbol{k}, \Omega)$: the Fourier transform of $U(\boldsymbol{x}, t)$. By a theorem of Paley and Wiener (see Reed and Simon 1975), if $U(\boldsymbol{x}, t)$ has support in the (shifted) light cone then, $\tilde{U}(\boldsymbol{k}\Omega)$ is analytic for $(k_2, \Omega_2) \in -\Gamma^*$. By explicit computation,

$$\tilde{U}(\boldsymbol{k}, \Omega) = -\mathrm{i}(2\pi)^{-1/2} \sum_{j=1}^{m} \frac{\tilde{h}(\boldsymbol{k})\, \mathrm{e}^{\mathrm{i}\Omega t_0}}{\Omega + \omega_j(\boldsymbol{k})}. \tag{4}$$

Specialising to $m = 1$ and $m = 2$:

$$\tilde{U}(\boldsymbol{k}, \Omega) = -\mathrm{i}(2\pi)^{-1/2} \frac{\tilde{U}_0(\boldsymbol{k})\, \mathrm{e}^{\mathrm{i}\Omega t_0}}{\Omega + \omega(\boldsymbol{k})}, \qquad m = 1, \tag{5}$$

$$\begin{aligned} \tilde{U}(\boldsymbol{k}, \Omega) = &-\mathrm{i}(2\pi)^{-1/2} \tilde{f}(k)\, \mathrm{e}^{\mathrm{i}\Omega t_0} \frac{\Omega + \omega_1(\boldsymbol{k}) + \omega_2(\boldsymbol{k})}{[\Omega + \omega_1(\boldsymbol{k})][\Omega + \omega_2(\boldsymbol{k})]} \\ &- (2\pi)^{-1/2} \tilde{g}(k)\, \mathrm{e}^{\mathrm{i}\Omega t_0} \frac{1}{[\Omega + \omega_1(\boldsymbol{k})][\Omega + \omega_2(\boldsymbol{k})]}, \qquad m = 2. \end{aligned} \tag{6}$$

$t_0 = R/c$, where R is the radius of a ball containing the initial data. By the Paley–Wiener theorem (see appendix) $t_0 < \infty$ is equivalent to $\tilde{U}_0(\boldsymbol{k})$, $\tilde{f}(\boldsymbol{k})$ and $\tilde{g}(\boldsymbol{k})$ being entire analytic.

To avoid repetitions the following are assumed throughout this section:

(a) $c, t_0 < \infty$;

(b) $\operatorname{Im} \omega_j(\boldsymbol{k}) \leq 0$, $\boldsymbol{k}$ real, all j (this involves no loss of generality if the medium is of at most exponential gain);

(c) $\omega_1(\boldsymbol{k}) \neq \omega_2(\boldsymbol{k})$ (a.e.).

$m = 1$

Lemma 3.1. $\omega(\boldsymbol{k})$ is entire analytic.

Proof. Fix $\boldsymbol{k}$ such that $W(\Omega) \equiv \tilde{U}(\boldsymbol{k}, \Omega)$ is not identically zero. $W(\Omega)$ is analytic at $(\boldsymbol{k}_2, \Omega_2) \in -\Gamma^*$ and so has discrete zeros in compacts. Inverting (5) gives $\Omega' + \omega(\boldsymbol{k}')$ jointly analytic, for suitable Ω', in a neighbourhood of $\boldsymbol{k}$. The completion of the argument is simplest in one dimension. $W(\Omega) \equiv 0$ for at most a discrete set of $\boldsymbol{k}$ and thus $\omega(\boldsymbol{k})$ has at most isolated singularities. But near an isolated singularity the denominator in (5) can be made zero in the dual cone, contradicting the analyticity of $\tilde{U}(\boldsymbol{k}, \Omega)$. Thus $\omega(\boldsymbol{k})$ is entire. □

Proposition 3.2.

$$c \geq \frac{\operatorname{Im} \omega(\boldsymbol{k})}{|\operatorname{Im} \boldsymbol{k}|} \tag{7}$$

where $|\operatorname{Im} \boldsymbol{k}|^2 = (\operatorname{Im} k_1)^2 + (\operatorname{Im} k_2)^2 + \ldots$.

Proof. (7) is the statement that $-\Omega + \omega(\boldsymbol{k})$ does not vanish for $(\boldsymbol{k}_2, \Omega_2) \in \Gamma^*$. Suppose the contrary. By lemma 3.1 there is an n-hypersurface S in Γ^* containing $(\boldsymbol{k}_2, \Omega_2)$ where $\Omega' = \omega(\boldsymbol{k})$, $(\boldsymbol{k}_2^1, \Omega_2^1) \in S$. Since $\omega(\boldsymbol{k})$ is differentiable at $\boldsymbol{k}$, the normal to S does not lie in the $\boldsymbol{k}_2$ hyperplane. Thus the projection of S on the $\boldsymbol{k}_2$ hyperplane contains an n-sphere about $\boldsymbol{k}_2$. But $\tilde{U}(\boldsymbol{k}, \Omega)$ is analytic in $-\Gamma^*$ and so $\tilde{U}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\Omega t_0}$ must vanish identically on the zeros of $\Omega + \omega(\boldsymbol{k})$. Thus $\tilde{U}_0(\boldsymbol{k})$ vanishes on a small n-sphere about $\boldsymbol{k}_2$. Since it is entire $\tilde{U}_0(\boldsymbol{k})$ vanishes identically. □

The proof of theorem 1 follows easily. By (7)

$$\operatorname{Im} \omega(\boldsymbol{k}) \leq c|\boldsymbol{k}|. \tag{8}$$

By a version of Liouville's theorem (Titchmarsh 1939), an entire function satisfying (8) is a polynomial of degree ≤ 1.

$m = 2$

For simplicity consider initial data $(0, \tilde{g}(\boldsymbol{k}))$.

The denominator in (6) is

$$\Delta(\boldsymbol{k}, \Omega) = \Omega^2 + \Omega[\omega_1(\boldsymbol{k}) + \omega_2(\boldsymbol{k})] + \omega_1(\boldsymbol{k})\omega_2(\boldsymbol{k}). \tag{9}$$

The analogue of lemma 3.1 is

Lemma 3.3. $\omega_1(\boldsymbol{k}) + \omega_2(\boldsymbol{k})$ and $\omega_1(\boldsymbol{k})\omega_2(\boldsymbol{k})$ are entire analytic.

The proof is essentially the same as in lemma 3.1.

Remark. In situations with time reversal invariance $\omega_1(\boldsymbol{k}) = -\omega_2(-\boldsymbol{k})$. If, in addition, $\omega(\boldsymbol{k})$ is isotropic the first condition holds trivially $(\omega_1(\boldsymbol{k})+\omega_2(\boldsymbol{k})=0)$ and one has only $\omega^2(\boldsymbol{k})$ entire.

Proposition 3.2 has the analogue:

Proposition 3.4.

$$c \geqslant \frac{\operatorname{Im} \omega_{1,2}(\boldsymbol{k})}{|\operatorname{Im} \boldsymbol{k}|}. \tag{10}$$

Proof. As before, (10) expresses $\Delta(\boldsymbol{k}, \Omega) \neq 0$ for $(\boldsymbol{k}_2, \Omega_2) \in -\Gamma^*$. Suppose first $2\Omega \neq \omega_1(\boldsymbol{k})+\omega_2(\boldsymbol{k})$. Then if $\Delta(\boldsymbol{k}, \Omega)=0$ the zero is simple, i.e. $\partial_\Omega \Delta(\boldsymbol{k}, \Omega) \neq 0$. This is analogous to the situation in proposition 3.2 and so one concludes that $\Delta(\boldsymbol{k}, \Omega)$ has no simple zeros. Consider now double points. Since $\tilde{g}(\boldsymbol{k})$ is assumed to be non-zero on double points, $\Delta(\boldsymbol{k}, \Omega)$ may not vanish for $(\boldsymbol{k}_2, \Omega_2) \in -\Gamma^*$. □

Remarks.

(1) $\partial(-\Gamma^*)$, the boundary of $-\Gamma^*$, may contain simple and double zeros of $\Delta(\boldsymbol{k}, \Omega)$.

(2) If $\omega_1(\boldsymbol{k})-\omega_2(\boldsymbol{k})$ has non-degenerate zeros, $\Delta(\boldsymbol{k}, \Omega)$ has simple zeros in a $(\boldsymbol{k}_2, \Omega_2)$ neighbourhood of double zeros. Theorem 3 then holds for any initial data.

Proposition 3.5.

(a) $\omega_1(\boldsymbol{k})+\omega_2(\boldsymbol{k}) = \boldsymbol{a} \cdot \boldsymbol{k} + b$, $\quad$ $\boldsymbol{a}$ real and $\operatorname{Im} b \leqslant 0$.

(b) $\omega_1^2(\boldsymbol{k})+\omega_2^2(\boldsymbol{k})$ is a polynomial in $\boldsymbol{k}$ of degree $\leqslant 2$.

Proof. By (10)

$$\operatorname{Im}[\omega_1(\boldsymbol{k})+\omega_2(\boldsymbol{k})] \leqslant 2c|\operatorname{Im} \boldsymbol{k}|. \tag{11}$$

Lemma 3.3 and Liouville's theorem give that $\omega_1(\boldsymbol{k})+\omega_2(\boldsymbol{k})$ is a polynomial of degree $\leqslant 1$. (c) of the 'standing assumptions' then gives (a). Write

$$\omega_1^2(\boldsymbol{k})+\omega_2^2(\boldsymbol{k}) = [\omega_1(\boldsymbol{k})+\omega_2(\boldsymbol{k})] - 2\omega_1(\boldsymbol{k})\omega_2(\boldsymbol{k}). \tag{12}$$

The LHS of (12) is entire by lemma 3.3. Now, by (a) and (10)

$$-c|\operatorname{Im} \boldsymbol{k}| + \operatorname{Im} b \leqslant \operatorname{Im} \omega(\boldsymbol{k}) \leqslant c|\operatorname{Im} \boldsymbol{k}|. \tag{13}$$

Hence

$$\begin{aligned} \operatorname{Re}[\omega_1^2(\boldsymbol{k})+\omega_2^2(\boldsymbol{k})] &\geqslant -[(\operatorname{Im} \omega_1(\boldsymbol{k}))^2 + (\operatorname{Im} \omega_2(\boldsymbol{k}))] \\ &\geqslant -2[c|\boldsymbol{k}| + 2c|\boldsymbol{k}| + b]. \end{aligned} \tag{14}$$

Invoking Liouville once again, for (14) now, proves (b). The theorem follows directly from proposition 3.5.

Acknowledgment

It is a pleasure to thank Professor R Seiler for useful conversations, Dr O Kafri for getting me interested in the question of causal dispersions and Professor J Rauch for useful correspondence.

Appendix

Liouville's Theorem (Titchmarsh 1939, p 87). Let $A(r)$ denote the upper bound on the real part of $f(z)$ on $|z| = r$. Then, if $f(z)$ is analytic for all finite z and $A(r) \leqslant Ar^k$ for $r \geqslant r_0$ with r_0, A and k constants, then $f(z)$ is a polynomial of degree $\leqslant k$.

Paley–Wiener Theorem (Reed and Simon 1975, p 23). Let T be a tempered distribution with support in the cone $\bar{\Gamma}$; then its Fourier transform $\tilde{T}$ extends as an analytic function to the tube $\mathbb{R}^n + \mathrm{i}\Gamma^*$.

References

Brillouin L 1960 *Wave Propagation and Group Velocity* (New York: Academic Press)
Krotschek E and Knudt W 1978 *Commun. Math. Phys.* **60** 171–80
Nussenzweig H M 1972 *Causality and Dispersion Relations* (New York: Academic Press)
Reed M and Simon B 1975 *Methods of Modern Mathematical Physics Vol II: Fourier Analytic Self-Adjointness* (New York: Academic Press)
Titchmarsh E C 1939 *The Theory of Functions* (Oxford: OUP)
Velo G and Wightman A S 1978 *Invariant Wave Equations* Springer Lecture notes, Berlin